§ 11.4 Notes on Mathematical Induction

Before we delve into the today's topic, let's review some basic set notation…

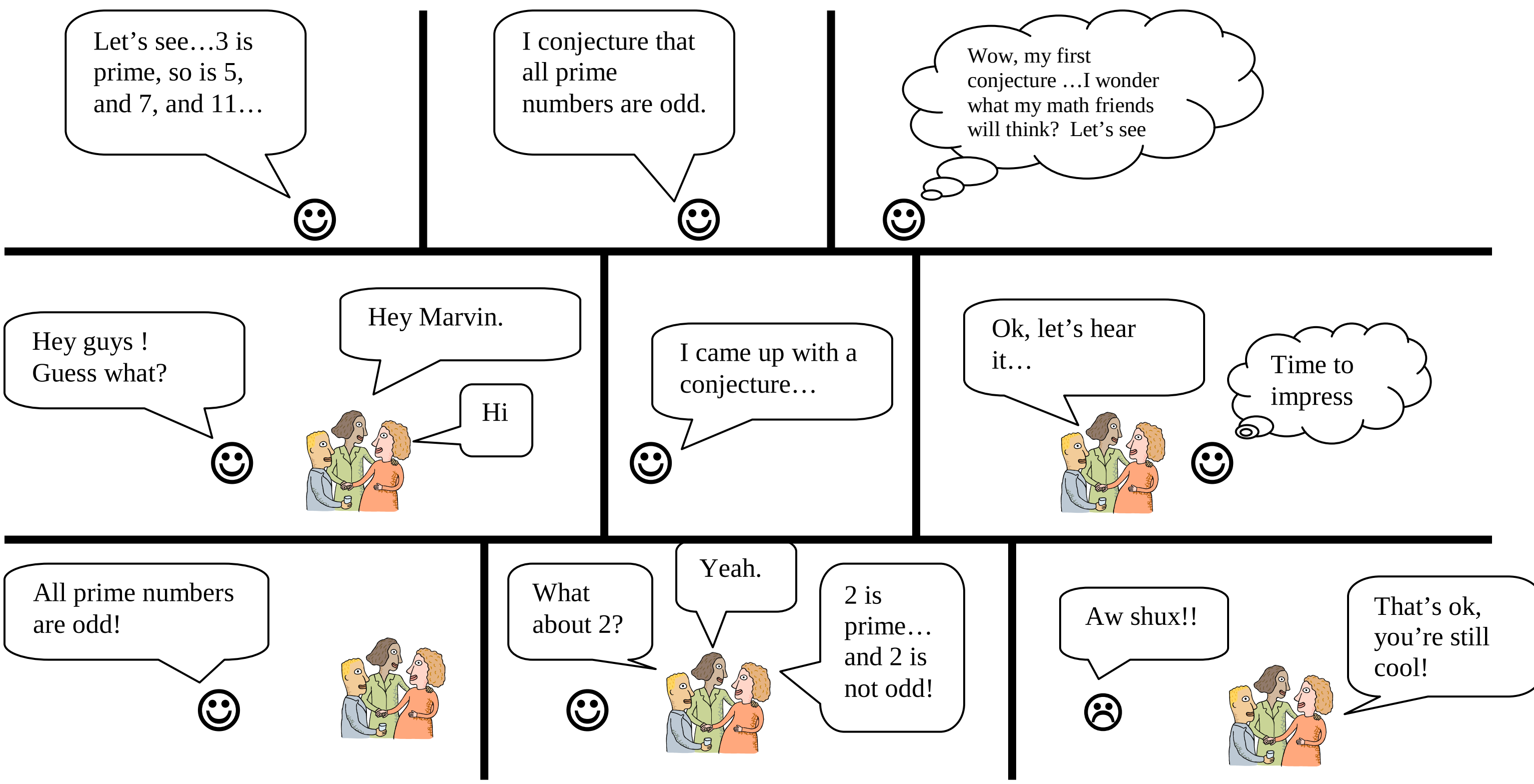
Set Notation Review

N – the set of positive integers (aka set of *natural numbers*) {1, 2, 3, …}
Z – the set of integers {…, -3, -2, -1, 0, 1, 2, 3, …}
Q – the set of rational numbers (written as a fraction a/b)
R – the set of real numbers

Notice how all sets are denoted by *CAPITAL LETTERS*. This is because lower case letters were already reserved for use by variables. Cool fact, huh?

Many important results in mathematics have been discovered by observing patterns in some specific cases and then by making generalizations from the observations.
→ This is called Applying INDUCTIVE REASONING. Mathematicians would then use their deductive reasoning skills to come up with rules…in fact, they were called ***conjectured rules*** (pretty fancy)

Now, mathematicians would give example after example proving that their "conjectured rule" is TRUE. Nice as this is, your friends only need one counterexample to prove that your conjecture is NOT TRUE.



Induction Principle: mathematicians use this idea to prove properties with *Natural* numbers. It works like this…

THE PRINCIPAL OF MATHEMATICAL INDUCTION

Let P_n be a statement involving the positive integer n.

If

1. P_1 is true for the natural number 1 (some books write this as P(1), or simply 1) AND
2. The truth of P_k implies the truth of P_{k+1}, for every positive k,

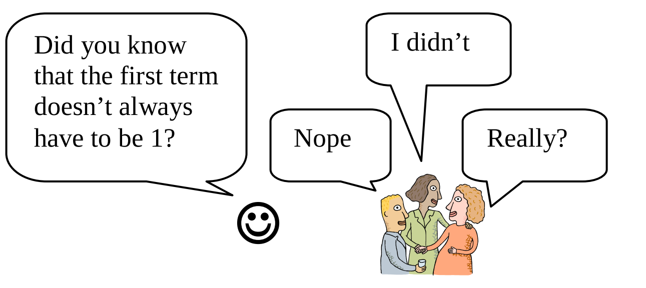
then P_n must be true for all positive integers n.

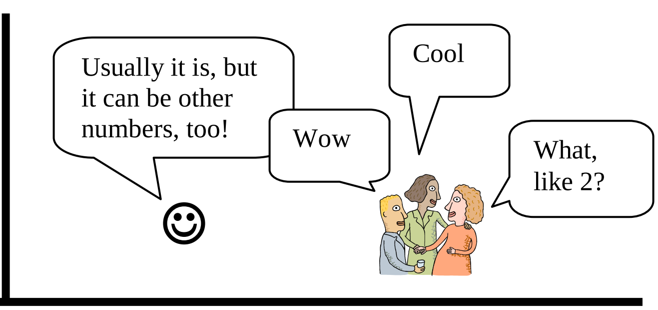
BOTH PARTS ARE NECESSARY TO USE
THE PRINCIPAL OF MATHEMATICAL INDUCTION

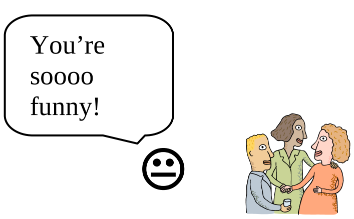
Now, there are two sides to every equation… and this is no different.

$$S_n = \underbrace{1 + 2 + 3 + \dots + n}_{\text{Term formula}} = \underbrace{\frac{n(n+1)}{2}}_{\text{Sum formula}}$$

The left side is called the term formula side, and the right is called the sum formula side.







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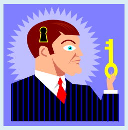
Speaking of terms, let's review what the next term is for given situations where *k* represents an integer…

Term	**Next term**	**Simplified version of next term**
k	(k + 1)	k + 1
k + 3	(k + 1) + 3	k + 4
k – 2	(k + 1) – 2	k – 1
2k + 3	2(k + 1) + 3	2k + 5
2^k	$2^{(k+1)}$	2^{k+1}
2^{k+1}	$2^{(k+1)+1}$	2^{k+2}
2^{k-1}	$2^{(k+1)-1}$	2^k
k(k + 1)	(k + 1)((k + 1) + 1)	(k + 1)(k + 2), ***or*** $k^2 + 3k + 2$
k^2	$(k+1)^2$	$k^2 + 2k + 1$

For geometric proofs, you needed *Statements* and *Reasons*. Mathematical Induction proofs need four parts:

- When *n* = 1,
- Assume P(*k*) is true for *k* ≥ 1,
- Then, and
- Therefore.



EX 1] Use mathematical induction to prove the statement is true for all natural numbers n.

$$S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Idea…when we get to the end of the THEN part… we want the right side to be in the form. Sometimes it is easier to work backwards if you get stuck.

$$\frac{(k+1)(k+2)}{2}$$

When $n = 1$

$1 = \frac{1(1+1)}{2}$ — (1) Write 1 on the left (since it is the first term), and substitute 1 for n on the sum formula (right) side

If the first term were 7, we'd put 7 on the left.

$1 = \frac{1(2)}{2}$ — (2) Simplify

$1 = 1$ — (3) P(1) is true!

Assume P(k) is true for $k \geq 1$. — (4) You have to actually state this

$S_k = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$ — (5) Replace n with k

mathematicians are soooo picky about how they write things!

Then, prove P($k + 1$) is true

$S_{k+1} = 1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$ — (6) Change S_k to S_{k+1}, add $(k + 1)$ to both sides

$= 1 + 2 + 3 + \ldots + k + (k+1) = (k+1)\left(\frac{k}{2}+1\right)$ — (7) Remember *Factor By Grouping*?

$= 1 + 2 + 3 + \ldots + k + (k+1) = (k+1)\left(\frac{k}{2}+\frac{2}{2}\right)$ — (8) Rewrite 1 as $\frac{2}{2}$

From (7) onward, you work with the right side

$= 1 + 2 + 3 + \ldots + k + (k+1) = (k+1)\left(\frac{k+2}{2}\right)$ — (9) Simplify

$= 1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$ — (10) Simplify

Therefore, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ is true for all $n \in$ N. $\therefore$ — (11) State conclusion

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EX 2] For this problem, we will look at two different approaches.
Use mathematical induction to prove $P_n = 2 + 4 + 6 + \ldots + 2n = n(n + 1)$ is true for all natural numbers.

When $n = 1$

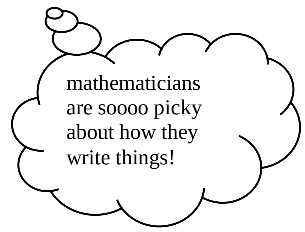
$2 = 1(1+1)$ (1) Write 2 on the left and 1 for n on the sum formula (right) side

$2 = 1(2)$ (2) Simplify

$2 = 2$ (3) P(1) is true!

Assume P(k) is true for $k \geq 1$. (4) You have to actually state this

$S_k = 2 + 4 + 6 + \ldots + 2k = k(k + 1)$ (5) Replace n with k



Then, prove P(k + 1) is true

$S_{k+1} = 2 + 4 + 6 + \ldots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$ (6)

$= S_k + 2(k + 1) = (k + 1)(k + 2)$ (7) You can substitute S_k for $2 + 4 + 6 + \ldots + 2k$ on the left and use *Factor By Grouping* on the right

Therefore, $2 + 4 + 6 + \ldots + 2n = n(n + 1)$ is true for all $n \in$ N. ∴ (8) State conclusion

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Example 2] $S_n = 2 + 4 + 6 + \ldots + 2n = n(n + 1)$ **Alternate Technique**

When $n = 1$

$2 = 1(1 + 1)$ (1) Write 2 on the left and 1 for n on the sum formula (right) side

$2 = 1(2)$ (2) Simplify

$2 = 2$ (3) P(1) is true!

Assume P(k) is true for $k \geq 1$. (4) You have to actually state this

$S_k = 2 + 4 + 6 + \ldots + 2k = k(k + 1)$ (5) Replace n with k

Then, prove P(k + 1) is true

$S_{k+1} = 2 + 4 + 6 + \ldots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$ (6)

$= S_k + 2(k + 1) = k^2 + k + 2k + 2$ (7) Substitute S_k and then distribute on the right side

$= S_k + 2(k + 1) = k^2 + 3k + 2$ (8) Simplify

$= S_k + 2(k + 1) = (k + 1)(k + 2)$ (9) Factor right side…and we're done!!

Therefore, $S_n = 2 + 4 + 6 + \ldots + 2n = n(n + 1)$ is true for all $n \in$ N. ∴ (10) State conclusion

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EX 3]

Use mathematical induction to prove $P_{\mathbf{n}} = 3 + 7 + 11 + 15 + \ldots + (4n - 1) = n(2n + 1)$ is true for all natural numbers.

Idea…when we get to the end of the THEN part…
we want the right side to be in the form $(k+1)(2(k+1)+1)$ or $(k+1)(2k+3)$

When $n = 1$

$3 = 1(2(1)+1)$	(1) Write 3 on the left and 1 for n on the sum formula (right) side
$3 = 1(3)$	(2) Simplify
$3 = 3$	(3) P(1) is true!

Assume P(k) is true for $k \geq 1$. (4) You have to actually state this

$S_k = 3 + 7 + 11 + 15 + \ldots + (4k - 1) = k(2k + 1)$ (5) Replace n with k

Then, prove P(k + 1) is true

$S_{k+1} = 3 + 7 + 11 + 15 + \ldots + (4k - 1) + (4(k+1) - 1) = k(2k + 1) + (4(k+1) - 1)$ (6)

$= S_k + (4k+3) = k(2k + 1) + (4k + 3)$ (7) Substitute S_k and then simplify $(4(k+1) - 1)$ to $(4k + 3)$

$= S_k + (4k+3) = 2k^2 + k + 4k + 3$ (8)

$= S_k + (4k+3) = 2k^2 + 5k + 3$ (9) Simplify

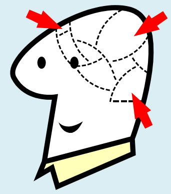
$= S_k + (4k+3) = (k+1)(2k + 3)$ (10) Factor

$= S_k + (4k+3) = (k+1)(2k + 2 + 1)$ (11) Split 3 into 2 + 1

$= S_k + (4k+3) = (k+1)(2(k + 1) + 1)$ (12)

Therefore, $3 + 7 + 11 + 15 + \ldots + (4n - 1) = n(2n + 1)$ is true for all $n \in$ N. ∴ (13) State conclusion

Sums of Powers of Integers:



- $\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + 3^4 + 4^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
- $\sum_{k=1}^{n} k^5 = 1^5 + 2^5 + 3^5 + 4^5 + \ldots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

EX 4] Find the sum using the formulas for the sums of powers of integers for $S_n = \sum_{n=1}^{5} n^4$

Solution: First, determine how many terms are in the sum … n = 1, 2, 3, 4, 5 → We have 5 terms

$$S_n = \sum_{n=1}^{5} n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$ (1) Write sum formula for n^4

$$= \frac{5((5)+1)(2(5)+1)(3(5)^2+3(5)-1)}{30}$$ (2) Substitute 5 for n

$$= \frac{5(6)(10+1)(75+15-1)}{30}$$ (3) Simplify

$$= \frac{30(11)(89)}{30}$$ (4) Simplify

$$= 979$$ (5) Simplify

*Verify on your graphing calculator.

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Math Analysis — Name: ______________________

§ 11.5 Notes on Mathematical Induction — Block: 1 2 3 4 5 6 7 8 — Date:

For geometric proofs, you needed *Statements* and *Reasons*. Mathematical Induction proofs need four parts:

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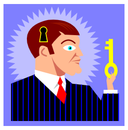


Use mathematical induction to prove the given statement is true for all natural numbers.

EX 1] $S_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

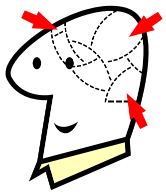
EX 2] $S_n = 2 + 4 + 6 + \ldots + 2n = n(n+1)$

EX 3] $S_n = 3 + 7 + 11 + 15 + \ldots + (4n-1) = n(2n+1)$

EX 4] Using the Sums of Powers of Integers formula, find the sum of $\sum_{k=1}^{5} k^4$.

Sums of Powers of Integers

- $\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \ldots + n = \dfrac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$
- $\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + 3^4 + 4^4 + \ldots + n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
- $\sum_{k=1}^{n} k^5 = 1^5 + 2^5 + 3^5 + 4^5 + \ldots + n^5 = \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$



<u>HW Problems:</u>

Use the Principal of Math Induction to prove that the given statement is true for all natural numbers, *n*.

1] $S_n = 1 + 4 + 7 + 10 + \ldots + (3n-2) = \dfrac{n}{2}(3n-1)$

2] $S_n = 1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$

3] $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

4] $S_n = 2 + 7 + 12 + \ldots (5n-3) = \dfrac{n}{2}(5n-1)$

Use the formulas for the sums of powers of integers find the sum of each.

5] $\sum_{k=1}^{6} k^2 - k$.

6] $\sum_{i=1}^{6} 6i^2 - 8i^3$

[#5 and #6 HINT: use 2 formulas!]

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Math Induction **HW Problems Key:**

1] $S_n = 1 + 4 + 7 + 10 + \ldots + (3n - 2) = \frac{n}{2}(3n-1)$

When $n = 1$

$1 = \frac{1}{2}(3(1)-1)$ (1) Write 1 on the left and 1 for *n* on the sum formula (right) side

$1 = \frac{1}{2}(2)$ (2) Simplify

$1 = 1$ (3) P(1) is true!

Assume P(k) is true for $k \geq 1$. (4) You have to actually state this

$S_k = 1 + 4 + 7 + 10 + \ldots + (3k - 2) = \frac{k}{2}(3k-1)$ (5) Replace *n* with *k*

Then, prove P(k + 1) is true

$S_{k+1} = 1 + 4 + 7 + 10 + \ldots + (3k - 2) + (3(k+1) - 2) = \frac{k}{2}(3k-1) + (3(k+1) - 2)$ (6)

$= S_k + (3(k+1) - 2) = \frac{k}{2}(3k-1) + (3k + 3 - 2)$ (7) Substitute S_k and then simplify

$= S_k + (3(k+1) - 2) = \frac{k}{2}(3k-1) + (3k + 1)$ (8)

$= S_k + (3(k+1) - 2) = \frac{k}{2}(3k-1) + \frac{2}{2}(3k+1)$ (9) Rewrite

$= S_k + (3(k+1) - 2) = \frac{1}{2}(k \bullet (3k-1) + 2(3k+1))$ (10) Factor out $\frac{1}{2}$

$= S_k + (3(k+1) - 2) = \frac{1}{2}(3k^2 - k + 6k + 2)$ (11)

$= S_k + (3(k+1) - 2) = \frac{1}{2}(3k^2 + 5k + 2)$ (12)

$= S_k + (3(k+1) - 2) = \frac{1}{2}((k+1)(3k+2))$ (13)

$= S_k + (3(k+1) - 2) = \frac{1}{2}((k+1)(3k+3-1))$ (14)

$= S_k + (3(k+1) - 2) = \frac{1}{2}((k+1)(3(k+1)-1))$ (15)

$= S_k + (3(k+1) - 2) = \frac{(k+1)}{2}((3(k+1)-1))$ (16)

Therefore, $1 + 4 + 7 + 10 + \ldots + (3n - 2) = \frac{n}{2}(3n-1)$ is true for all $n \in$ N. $\therefore$ (17) State conclusion

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2] $S_n = 1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$

When $n = 1$	$1 = 2^1 - 1$	(1) Write 1 on the left and 1 for *n* on the sum formula (right) side
	$1 = 2 - 1$	(2) Simplify
	$1 = 1$	(3) P(1) is true!

Assume P(*k*) is true for $k \geq 1$. — (4) You have to actually state this

$S_k = 1 + 2 + 2^2 + \ldots + 2^{k-1} = 2^k - 1$ — (5) Replace *n* with *k*

Then, prove P(*k* + 1) is true

$S_{k+1} = \underline{1 + 2 + 2^2 + \ldots + 2^{k-1}} + 2^k = 2^k - 1 + 2^k$ (6)

$= S_k + 2^k = 2^k - 1 + 2^k$ — (7) Substitute S_k

$= S_k + 2^k = (2)2^k - 1$ — (8) Simplify

$= S_k + 2^k = 2^{k+1} - 1$ — (9) Remember, same base for $(2)2^k$, so add exponents

Therefore, $S_n = 1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$ is true for all $n \in$ N. ∴ — (10) State conclusion

3] $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

When n = 1	$1^2 = \dfrac{1(1+1)(2(1)+1)}{6}$	(1) Write 1 on the left and 1 for n on the sum formula (right) side
	$1 = \dfrac{1(2)(3)}{6}$	(2) Simplify
	$1 = 1$	(3) P(1) is true!

Assume P(k) is true for $k \geq 1$.

(4) You have to actually state this

$S_k = 1^2 + 2^2 + 3^2 + \ldots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$ (5) Replace n with k

Then, prove P(k + 1) is true

$S_{k+1} = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ (6)

$= S_k + (k+1)^2 = \dfrac{k(k+1)(2k+1)}{6} + \dfrac{6(k+1)^2}{6}$ (7) Substitute S_k and get common denominator

$= S_k + (k+1)^2 = \dfrac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$ (8) Factor out (k + 1)

$= S_k + (k+1)^2 = \dfrac{(k+1)\left[2k^2+k+6k+6\right]}{6}$ (9) Distribute

$= S_k + (k+1)^2 = \dfrac{(k+1)\left[2k^2+7k+6\right]}{6}$ (10) Simplify

$= S_k + (k+1)^2 = \dfrac{(k+1)(k+2)(2k+3)}{6}$ (11) ***Could*** stop here and go to (14)

$= S_k + (k+1)^2 = \dfrac{(k+1)((k+1)+1)+(2k+2+1)}{6}$ (12)

$= S_k + (k+1)^2 = \dfrac{(k+1)((k+1)+1)+(2(k+1)+1)}{6}$ (13)

Therefore, $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ is true for all $n \in$ N. $\therefore$ (14) State conclusion

4] $S_n = 2 + 7 + 12 + \ldots (5n - 3) = \frac{n}{2}(5n-1)$

When $n = 1$	$2 = \frac{1}{2}(5(1)-1)$	(1) Write 1 on the left and 1 for n on the sum formula (right) side
	$2 = \frac{1}{2}(4)$	(2) Simplify
	$2 = 2$	(3) P(1) is true!

Assume P(k) is true for $k \geq 1$. (4) You have to actually state this

$S_k = 2 + 7 + 12 + \ldots (5k - 3) = \frac{k}{2}(5k-1)$ (5) Replace n with k

Then, prove P(k + 1) is true

$S_{k+1} = 2 + 7 + 12 + \ldots (5k - 3) + (5(k + 1) - 3) = \frac{k}{2}(5k-1) + (5(k + 1) - 3)$ (6)

$= S_k + (5k + 2) = \frac{k}{2}(5k-1) + (5k + 2)$ (7) Substitute S_k and simplify

$= S_k + (5k + 2) = \frac{k}{2}(5k-1) + \frac{2(5k+2)}{2}$ (8) Get common denominator

$= S_k + (5k + 2) = \frac{5k^2 - k + 10k + 4}{2}$ (9)

$= S_k + (5k + 2) = \frac{5k^2 + 9k + 4}{2}$ (10)

$= S_k + (5k + 2) = \frac{(k+1)(5k+4)}{2}$ (11)

$= S_k + (5k + 2) = \frac{(k+1)(5(k+1)-1)}{2}$ (12)

Therefore, $S_n = 2 + 7 + 12 + \ldots (5n - 3) = \frac{n}{2}(5n-1)$ is true for all $n \in$ N. ∴ (13) State conclusion

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Use the formulas for the sums of powers of integers find the sum of each.

5] $\sum_{k=1}^{6} k^2 - k = \frac{6(6+1)(2(6)+1)}{6} - \frac{6(6+1)}{2} = \frac{6(7)(13)}{6} - \frac{6(7)}{2} = 91 - 21 = 70$

You have to show the substitution into the formulas to receive credit!!!

6] $\sum_{i=1}^{6} 6i^2 - 8i^3 = 6\left(\frac{6(6+1)(2(6)+1)}{6}\right) - 8\left(\frac{(6)^2(6+1)^2}{4}\right) = 6(7)(13) - \frac{8(36)(49)}{4} = 546 - 3528 = -2982$

Still completing this part…

EX 5] Prove $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

When $n = 1$

$1^4 = \frac{1(1+1)(2 \bullet 1+1)(3 \bullet 1^2 + 3 \bullet 1 - 1)}{30}$ (1) $1^4 = 1$ (left side), substitute on right side

$1 = \frac{1(2)(3)(5)}{30}$ (2) Simplify

$1 = \frac{30}{30}$ (3) Simplify

$1 = 1$ (3) P(1) is true!

Assume S(k) is true for $k \geq 1$. (4) State this

$S_k = \sum_{k=1}^{n} k^4 = 1^4 + 2^4 + 3^4 + 4^4 + \ldots + k^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$ (5) Replace n with k

Then, prove P(k + 1) is true

$S_{k+1} = 1^4 + 2^4 + 3^4 + 4^4 + \ldots + k^4 + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4$ (6)

$= S_k + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4$ (7) Substitute

$= S_k + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + \frac{30(k+1)^4}{30}$ (8) LCD

$= S_k + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1) + 30(k+1)^4}{30}$ (9) Simplify

$= S_k + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1) + 30(k+1)^4}{30}$ (10) Simplify

$= S_k + (k+1)^4 = \frac{(k+1)[k(2k+1)(3k^2+3k-1) + 30(k+1)^3]}{30}$ (11) Factor By Grouping

$$= S_k + (k+1)^4 = \frac{(k+1)[6k^4 + 39k^3 + 91k^2 + 89k + 30]}{30}$$ (12) Simplify

You can thank me later…

$$= S_k + (k+1)^4 = \frac{(k+1)[(2k^2 + 7k + 6)(3k^2 + 9k + 5)]}{30}$$ (13) Factor

$$= S_k + (k+1)^4 = \frac{(k+1)[(k+2)(2k+3)(3k^2 + 9k + 5)]}{30}$$ (13) Factor $2k^2 + 7k + 6$

$$= S_k + (k+1)^4 = \frac{(k+1)(k+2)[(2(k+1)+1))(3k^2 + 9k + 5)]}{30}$$ (14)Rewrite 2k + 3 as 2(k + 1) + 1

$$= S_k + (k+1)^4 = \frac{(k+1)(k+2)[(2(k+1)+1))(3(k+1)^2 + 9(k+1) - 1)]}{30}$$ (15)

In (15), we replace every *k* with (*k* + 1).
Then, use your algebra skills to determine what number goes here

$3k^2 + 9k + 5 \quad = 3(k + 1)^2 + 9(k + 1) +$ _______________
$= 3(k^2 + 2k + 1) + 9k + 9 +$ _______________
$= 3k^2 + 6k + 3 + 9k + 9 +$ _______________
$= 3 k^2 + 15k + 12$