§ 11.5 Notes on The Binomial Theorem

Open your text to page 816 to complete the following section:

$(x + a)^{1}$ = ______________________________

$(x + a)^{2}$ = ______________________________

$(x + a)^{3}$ = ______________________________

$(x + a)^{4}$ = ______________________________

NOTE: Observations about these expansions:

1) In each expansion there are __________ terms.

2) In each expansion, *a* and *b* have _______________ roles.
The powers of *a* decrease by 1 in successive terms, whereas the powers of *b* increase by 1.

3) The sum of the powers of each term is _____.
EX] In the expansion of $(x + y)^{5}$, the sum of the powers of each term is _____.

4)The coefficients ____________________________________.

Binomial Theorem:

If x and a are real numbers and n is a positive integer, then

$$(x + a)^{n} = \binom{n}{0}x^{n} + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^{2} + \ldots + \binom{n}{n-1}xa^{n-1} + \binom{n}{n}a^{n}$$

$$= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^{j}$$

where $\binom{n}{j}$ is the ***binomial coefficient*** $\binom{n}{j} = \dfrac{n!}{j!\,(n-j)!}$.

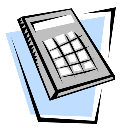
Remember: Factorial Notation: $n! = n\,(n-1)(n-2)\ldots 3\bullet 2\bullet 1$

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

This is read as "*n over j*" and represents the ***binomial coefficient***.

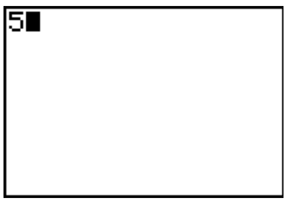
Note: Our textbook notates this as ${}_{n}C_{j}$, while most other texts use the form ${}_{n}C_{r}$. In the formula, each j would then be replaced with an r.

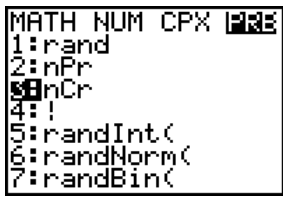
TIP

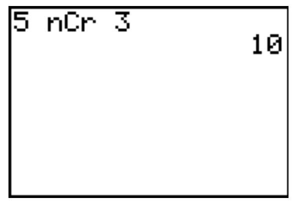


Your calculator can find it by the following:

$\binom{5}{3}$ would be 5 MATH ▸ PRB 3:nCr ENTER 3 ENTER

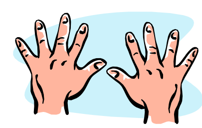






Your calculator then displays your answer. Isn't technology great!☺

You'll still have to be able to do this by hand, because your calculator can't handle really big numbers



- Open your text to page 816 and copy down **Example 1.**
- Be sure to use your calculator for (d).
- Copy down all steps, as this is what is expected on your homework problems as well as on a test/quiz to receive credit.
- Copy down the four useful formulas found on page 817 into your notes.

You've Got Problems!

Complete by hand, and verify with your calculator.

1) Find $\binom{5}{3}$

3) Find ${}_{6}C_{4}$

2) Find $\binom{15}{7}$

4) Find ${}_{12}C_{11}$

Solutions:

1) $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!} = \frac{5\bullet4\bullet\cancel{3!}}{\cancel{3!}(2)!} = \frac{20}{2} = 10$

2) $\binom{15}{7} = \frac{15!}{7!(8)!} = \frac{15\bullet14\bullet13\bullet12\bullet11\bullet10\bullet9\bullet\cancel{8!}}{7\bullet6\bullet5\bullet4\bullet3\bullet2\bullet1\bullet\cancel{(8)!}} = 6435$

3) $_6C_4 = \binom{6}{4} = \frac{6!}{4!(2)!} = \frac{6\bullet5\bullet\cancel{4!}}{\cancel{4!}(2)!} = \frac{30}{2} = 15$

4) $_{12}C_{11} = \binom{12}{11}$ uses the formula $\binom{n}{n-1} = n$ → $_{12}C_{11} = 12$

EXPANDING BINOMIALS

There are two methods to use to expand binomials – one is Pascal's Triangle, and the other is the Binomial Theorem. We will take a look at each…

Pascal's Triangle Method

Expand $(2a + b)^3$ using Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
. . . . . .

(1) List the coefficients of the correct row in the triangle…the *degree* tells you which row!

$(2a + b)^3 = 1(\ )(\ ) + 3(\ )(\ ) + 3(\ )(\ ) + 1(\ )(\ )$

(2) Fill in the terms

$(2a + b)^3 = 1(2a)(b) + 3(2a)(b) + 3(2a)(b) + 1(2a)(b)$

(3) Fill in the exponents

$(2a + b)^3 = 1\overbrace{(2a)^3(b)^0}^{3+0\,=\,3} + 3\overbrace{(2a)^2(b)^1}^{2+1=\,3} + 3\overbrace{(2a)^1(b)^2}^{1+2\,=\,3} + 1\overbrace{(2a)^0(b)^3}^{0+3\,=\,3}$

Notice how all exponent pairs add up to 3…

(4) Simplify

$(2a + b)^3 = 1(8a^3)(1) + 3(4a^2)(b) + 3(2a)(b^2) + 1(1)(b^3)$

(5) Simplify

$(2a + b)^3 = 8a^3 + 12a^2b + 6ab^2 + b^3 \therefore$

Binomial Theorem Method

Expand $(2a + b)^{\mathbf{3}}$ using the Binomial Theorem

This method is similar to using *Pascal's Triangle Method*, except that you have to compute the binomial coefficients, whereas in *Pascal's Triangle Method*, the row gives you the binomial coefficients.

Notice the top # is always 3…

(1) $$(2a + b)^{\mathbf{3}} = \binom{3}{0}(2a)^3(b)^0 + \binom{3}{1}(2a)^2(b)^1 + \binom{3}{2}(2a)^1(b)^2 + \binom{3}{3}(2a)^0(b)^3$$

…and notice how the bottom # begins at 0 and increases to 3.

(2) $$(2a + b)^{\mathbf{3}} = \left(\frac{3!}{0!(3-0)!}\right)(2a)^3(b)^0 + \left(\frac{3!}{1!(3-1)!}\right)(2a)^2(b)^1 + \left(\frac{3!}{2!(3-2)!}\right)(2a)^1(b)^2 + \left(\frac{3!}{3!(3-3)!}\right)(2a)^0(b)^3$$

(3) $$(2a + b)^{\mathbf{3}} = \left(\frac{3!}{0!(3)!}\right)(2a)^3(b)^0 + \left(\frac{3!}{1!(2)!}\right)(2a)^2(b)^1 + \left(\frac{3!}{2!(1)!}\right)(2a)^1(b)^2 + \left(\frac{3!}{3!(0)!}\right)(2a)^0(b)^3$$

(4) $$(2a + b)^{\mathbf{3}} = \left(\frac{\cancel{3!}}{0!(\cancel{3})!}\right)(2a)^3(b)^0 + \left(\frac{3 \bullet \cancel{2!}}{1!(\cancel{2})!}\right)(2a)^2(b)^1 + \left(\frac{3 \bullet \cancel{2!}}{\cancel{2!}(1)!}\right)(2a)^1(b)^2 + \left(\frac{\cancel{3!}}{\cancel{3!}(0)!}\right)(2a)^0(b)^3$$

(5) $$(2a + b)^{\mathbf{3}} = 1(2a)^3(b)^0 + 3(2a)^2(b)^1 + 3(2a)^1(b)^2 + 1(2a)^0(b)^3$$

* This step is the same as (2) and (3) in *Pascal's Triangle Method*

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

(6) $$(2a + b)^{\mathbf{3}} = 1(8a^3)(1) + 3(4a^2)(b) + 3(2a)(b^2) + 1(1)(b^3)$$

(7) $$(2a + b)^{\mathbf{3}} = 8a^3 + 12a^2b + 6ab^2 + b^3 \;\therefore$$

Note:

When you are asked to expand using the binomial method, your work needs to include how you set up the problem, as illustrated above.

FINDING INDIVIDUAL TERMS

Sometimes you will be asked to find a specific term in a binomial expansion, not needing to expand the whole thing! Write down each below. Describe any patterns you notice in the set up. Then find the term asked for.

Problem	Set up
1. Find the 4th term of $(x + 3y)^{11}$	$\binom{11}{3}(x)^{11-3}(3y)^3$
2. Find the 7th term of $(4x - 2y)^{15}$	$\binom{15}{6}(4x)^{15-6}(-2y)^6$
3. Find the 10th term of $\left(\frac{1}{2}x+3y\right)^{12}$	$\binom{12}{9}\left(\frac{1}{2}x\right)^{12-9}(3y)^9$

Solutions:

Description of patterns for finding the 4th term of $(x + 3y)^{11}$:

The degree the binomial is being expanded to

$$\binom{11}{3}(x)^{11-3}(3y)^3$$

One less than the term asked for.

1. $$\binom{11}{3}x^{11-3}(3y)^3=\frac{11!}{3!(8)!}(x^8)(27y^3)=\frac{11\bullet 10\bullet 9\bullet 8!}{3\bullet 2\bullet 1\bullet (8)!}(x^8)(27y^3)=165(x^8)(27y^3)=4455x^8y^3$$

1 point

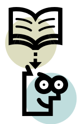
1 point

Point values for this type of problem on a quiz/test

2 points

2. $$\binom{15}{6}(4x)^{15-6}(-2y)^6=\frac{15!}{6!(9)!}(4x)^9(-2y)^6=(5005)\left(262144x^9\right)\left(64y^6\right)=8.396996608\times 10^{10}x^9y^6$$

3. $$\binom{12}{9}\left(\frac{1}{2}x\right)^{12-9}(3y)^9=\frac{12!}{9!(3)!}\left(\frac{1}{2}x\right)^3(3y)^9=\frac{12\bullet 11\bullet 10\bullet 9!}{9!(3)!}\left(\frac{1}{8}x^3\right)(19683y^9)=541282.5x^3y^9$$



Create your own Problem and Set it Up

Pair up with someone close to you. Have that person complete the appropriate column below and sign their name.

I agree with my partner's example because	**I disagree with my partner's example because**
Signed: ______________________	**Signed:** ______________________

You've Got Problems!

5) Expand $(2x - 3y)^5$ using *Pascal's Triangle Method.*

6) Expand $(3a + b)^3$ using the *Binomial Theorem Method.*

7) Find the fourth term of $(2a - 6b)^{11}$.

8) What is the coefficient of x^8y^7 in the expansion of $(x + y)^{15}$?

You've Got HW Problems!

Pg. 821 #5 – 23 (o), 33, 35, 37
Quiz 11.1-11.3

Solutions:

5) Expand $(2x – 3y)^5$ using *Pascal's Triangle Method.*

$$(2x-3y)^5 = 1(2x)^5(3y)^0 - 5(2x)^4(3y)^1 + 10(2x)^3(3y)^2 - 10(2x)^2(3y)^3 + 5(2x)^1(3y)^4 - 1(2x)^0(3y)^5$$
$$= 32x^5(1) - 5(16x^4)(3y) + 10(8x^3)(9y^2) - 10(4x^2)(27y^3) + 5(2x)(81y^4) - 1(1)(243y^5)$$
$$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$$

6) Expand $(3a + b)^3$ using the *Binomial Theorem Method.*

$$(3a+b)^{\mathbf{3}} = \binom{3}{0}(3a)^3(b)^0 + \binom{3}{1}(3a)^2(b)^1 + \binom{3}{2}(3a)^1(b)^2 + \binom{3}{3}(3a)^0(b)^3$$

1 point

$$= (1)(27a^3)(1) + 3(9a^2)(b) + 3(3a)(b^2) + (1)(1)(b^3)$$
$$= 27a^3 + 27a^2b + 9ab^2 + b^3$$

1 point

7) Find the fourth term of $(2a – 6b)^{\mathbf{11}}$.

$$\binom{11}{3}(2a)^{11-3}(-6b)^3 = 165(2a)^8(-6b)^3 = -9123840a^8b^3$$

1 point 1 point

8) What is the coefficient of $x^{\mathbf{8}}y^{\mathbf{7}}$ in the expansion of $(x + y)^{\mathbf{15}}$?

$x^{\mathbf{8}}y^{\mathbf{7}}$ **tells us we want row 8…** $\binom{15}{7}x^8y^7 = 6435$