Section 3.5

Date__________

Polynomial & Rational Inequalities

To solve a Polynomial or Rational Inequality:

1. Rewrite the inequality so that all terms are on the left and zero is on the RIGHT.

(Make sure that rational expressions are written As one Quotient / Fraction.)

2. Find the zeros of the expression on the left and/or where it is undefined.
3. Use the zeros and undefined values to break the x-axis into intervals.
4. Choose a value for x in each interval and evaluate it in the expression on the left.
   - Positive values mean that $f(x) > 0$.
   - Negative values mean that $f(x) < 0$.

★ **Strict** inequalities (< or >) will use open intervals, which means the endpoints of the intervals are not solutions; whereas **not-strict** inequalities ($\le$ or $\ge$) will use closed intervals, which means that the endpoints are solutions.

#12

Ex 1 Solve the inequality. $6x^2 \le 6 + 5x$

$6x^2 - 5x - 6 \le 0$

$(2x-3)(3x+2) \le 0$

$2x-3=0$ $3x+2=0$

$x = 3/2$ $x = {}^-2/3$

$+ \quad - \quad +$

$-4/3 \quad 3/2$

Answer!

$\boxed{[-2/3,\ 3/2]}$

Test

$x=-1$	$x=0$	$x=2$
$(2(-1)-3)(3(-1)+2)$	$(-3)(2)$	$(2\cdot2-3)(3\ 2+2)$
$(-5)(-1)$	-6	$(1)(8)$
5		8

Ex 2 Solve the inequality. $x^3 + 2x^2 - 3x > 0$ #24

$x(x^2+2x-3) > 0$

$x(x-1)(x+3) > 0$

$x=0$ $x-1=0$ $x+3=0$

$x=1$ $x={}^-3$

$- \quad + \quad - \quad +$

$-3 \quad 0 \quad 1$

$x=-4$	$x=-2$	$x=\frac{1}{2}$	$x=2$
$(-)(-)(-)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

$\boxed{(-3,0) \cup (1,\infty)}$

Ex 3 Solve the inequality. $\dfrac{x^2-x-6}{x-1} \le 0$

$\dfrac{(x-3)(x+2)}{x-1} \le 0$

$x-3=0$ $x+2=0$ $x-1=0$

$x=3$ $x={}^-2$ $x=1$

$- \quad + \quad - \quad +$

$-2 \quad 1 \quad 3$

$x=-4$	$x=0$	$x=2$	$x=4$
$\dfrac{(-)(-)}{(-)}$	$\dfrac{(-)(+)}{(-)}$	$\dfrac{(-)(+)}{(+)}$	$\dfrac{(+)(+)}{(+)}$

$\boxed{(-\infty,-2] \cup (1,3]}$

#44

Ex 4 Solve the inequality. $\frac{5}{x-3} > \frac{3}{x+1}$

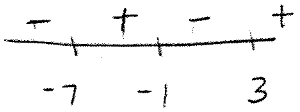
$\frac{5}{x-3} - \frac{3}{x+1} > 0$

$\frac{5(x+1)}{(x-3)(x+1)} - \frac{3(x-3)}{(x+1)(x-3)} > 0$

$\frac{(5x+5)-(3x-9)}{(x-3)(x+1)} > 0$

$\frac{2x+14}{(x-3)(x+1)} > 0$

$\frac{2(x+7)}{(x-3)(x+1)} > 0$



$x=-8$	$x=-6$	$x=0$	$x=4$
$\frac{2(-)}{(-)(-)}$	$\frac{2(+)}{(-)(-)}$	$\frac{2(+)}{(-)(+)}$	$\frac{2(+)}{(+)(+)}$

$\boxed{(7,-1) \cup (3,\infty)}$

WATCH BRACKETS! − Symbol tells you to use soft brackets

#58

Ex 5 A ball is thrown vertically upward with an initial velocity of 96 feet per second. The distance s (in feet) of the ball from the ground after t seconds is $s = 96t - 16t^2$. For what time interval is the ball more than 112 feet above the ground?

Note: You could also solve using $-16(t^2-6t+7)$

$96t - 16t^2 > 112$

$-16t^2 + 96t - 112 > 0$

Solve using Quad. Formula:

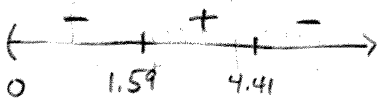
$A = -16$
$B = 96$
$C = -112$

$t = \frac{-96 \pm \sqrt{96^2 - 4(-16)(-112)}}{2(-16}$

$t = \frac{-96 \pm \sqrt{2048}}{-32}$

$t = \frac{-94 + \sqrt{2048}}{-32} \approx 1.59$

$t = \frac{-96 - \sqrt{2048}}{-32} \approx 4.41$



The ball is more than 112 ft above the ground after 1.59 sec. and before 4.41 sec.

* Verify Graphically *

#60

Ex 6 The monthly revenue achieved by selling x boxes of candy is calculated to be x(5 - 0.05x) dollars. The wholesale cost of each box of candy is $1.50. How many boxes must be sold each month to achieve a profit of at least $60?

Profit = Revenue - Cost

$P(x) = x(5-0.05x) - 1.50x$

$P(x) = 5x - 0.05x^2 - 1.50x$

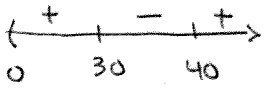
$P(x) = -0.05x^2 + 3.5x$

$-0.05x^2 + 3.5x \geq 60$

$-0.05x^2 + 3.5x - 60 \geq 0$

$x^2 - 70x + 1200 \leq 0$ ← (DIVIDE BY −0.05)

$(x-40)(x-30) \leq 0$



The profit is at least $60 when 30 to 40 boxes are sold.

Section 3.6 - Day 1

Date________

The Real Zeros of a Polynomial Function

The Remainder Theorem: Let f be a polynomial function. If f(x) is divided by x - c, then the remainder is f(c).

The Factor Theorem: Let f be a polynomial function. Then x - c is a factor of f(x) if and only if f(c) = 0.

1. If f(c) = 0, then x - c is a factor of f(x).
2. If x - c is a factor of f(x), then f(c) = 0.

Note: if sum of coeff. = 0, then x = 1 is a zero of polynomial!

Ex 1 Use the Remainder or Factor Theorem to find the following.

a) Find the remainder if $f(x) = 2x^4 - 5x^2 + x - 1$ is divided by x + 3. -3 | 2 0 -5 1 -1 -6 18 -39 114 2 -6 13 -38 113 THE REMAINDER IS $\frac{113}{x+3}$	b) Is $x + \frac{1}{4}$ a factor of $f(x) = 4x^5 + x^4 - 12x - 3$? $-\frac{1}{4}$ | 4 1 0 0 -12 -3 ↓ -1 0 0 0 3 4 0 0 0 -12 0 since $f(-\frac{1}{4}) = 0$, $(x+\frac{1}{4})$ is a factor of f(x). Note: Don't forget place holders!!	c) Is x + 3 a factor of $f(x) = x^3 - 5x + 12$? -3 | 1 0 -5 12 -3 9 -12 1 -3 4 | 0 Yes, (x+3) is a factor of f(x) since f(-3) = 0

Number of Real Zeros: A polynomial function cannot have more real zeros than its degree.

Descartes' Rule of Signs: Let f be a polynomial function written in standard form.

- The number of **positive real zeros** of f either equals the number of variations in the sign of the nonzero coefficients of f(x) or else equals that number less an even integer.
- The number of **negative real zeros** if f either equals the number of variations in the sign of the nonzero coefficients of f(-x) or else equals that number less an even integer.

Ex 2 Tell the maximum number of real zeros of the polynomial. Then use Descartes' Rule of signs to find how many positive and negative zeros the polynomial may have.

$f(x) = -3x^5 + 4x^4 + 2$

max # zeros = 5, based on DEGREE.

Descartes Rule:

$f(x) = \underbrace{-3x^5 + 4x^4}_{} + 2$

$\ominus$ to $\oplus$ sign CHANGE

1 REAL POS. ZERO

$f(-x) = -3(-x)^5 + 4(-x)^4 + 2$

$f(-x) = 3x^5 + 4x^4 + 2$

No sign CHANGES

Max # of Real Zeros: 5   # Positive Zeros: 1   # Negative Zeros: 0

Rational Zeros Theorem: Let f be a polynomial function of degree 1 or higher in standard form where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then **p must be a factor** of the constant of the polynomial and **q must be a factor** of the leading coefficient of the polynomial.

Ex 3 List the possible rational zeros of the function below, then find the rational zeros. Write the function in factored form.

$f(x) = 2x^3 + 5x^2 - x - 6$ Possible Zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

$\underset{q}{\uparrow}$ $\underset{p}{\uparrow}$

FACTORS OF 6	±1	±2	±3	±6
FACTORS OF 2	±1	±2		

$$x=1 \;\begin{array}{|rrrr} 2 & 5 & -1 & -6 \\ & 2 & 7 & 6 \\ \hline 2 & 7 & 6 & \big| \; 0 \end{array}$$

$2x^2+7x+6=0 \rightarrow (2x+3)(x+2)=0$

$2x+3=0$ $x+2=0$

$x=-3/2$ $x=-2$

Rational Zeros: $x=1, -3/2, -2$ Factored Form of f(x) = $(x-1)(2x+3)(x+2)$

#68 Ex 4 Solve the following polynomial equation in the real number system.

$2x^4 + x^3 - 24x^2 + 20x + 16 = 0$ At most → 4 real zeros

② sign changes → 2 real zeros are ⊕ or none

$f(-x) = 2(-x)^4 + (-x)^3 - 24(-x)^2 + 20(-x) + 16$

$f(-x) = 2x^4 - x^3 - 24x^2 - 20x + 16$

2 sign changes → 2 negative zeros or none

$\begin{matrix} p \rightarrow \\ q \rightarrow \end{matrix} \frac{16}{2} \Rightarrow \frac{\pm 1, \pm 2, \pm 4, \pm 16}{\pm 1, \pm 2} \rightarrow \frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 16, \pm \frac{1}{2}, \pm 8$

$$2 \;\begin{array}{|rrrrr} 2 & 1 & -24 & 20 & 16 \\ & 4 & 10 & -28 & -16 \\ \hline 2 & 5 & -14 & -8 & \big| \; 0 \end{array}$$

$$-4 \;\begin{array}{|rrrr} 2 & 5 & -14 & 8 \\ & -8 & 12 & 8 \\ \hline 2 & -3 & -2 & \big| \; 0 \end{array}$$

$2x^2 - 3x - 2 = 0$

$(2x+1)(x-2)=0$

$x = -1/2$ $x = 2$

THE FACTORS ARE $f(x) = (x-2)(x+4)(2x+1)(x-2)$

$\therefore f(x) = (x-2)^2(x+4)(2x+1)$

The zeros Are $x=2$, $x=-4$, $x=-1/2$

Intermediate Value Theorem: Let f denote a polynomial function. If a < b and if f(a) and f(b) are opposite in sign, then there is at least one zero of f between a and b.

Ex 3 Show that the function $f(x) = x^4 + 8x^3 - x^2 + 2$ has a zero in the interval [-1, 0].

$f(-1) = (-1)^4 + 8(-1)^3 - (-1)^2 + 2$

$f(-1) = 1 - 8 - 1 + 2$

$f(-1) = -6$ (negative)

$f(0) = 2$ (positive)

Since f(x) changes from negative to positive in [-1, 0], there is at least one zero in [-1, 0].

Ex 4 Given a solution r that is in the interval $-3 \leq r \leq -2$, use the Intermediate Value Theorem to approximate the solution of $3x^3 - 10x + 9 = 0$.

① $f(-3) = -42$

$f(-2) = 5$

② DIVIDE INTO 10 = Subintervals between [-3, -2]

③ Evaluate at ENDPOINTS UNTIL You GET A Sign CHANGE

$\left.\begin{array}{l} f(-2.1) = 2.217 \\ f(-2.2) = {}^{-}0.944 \end{array}\right\}$ CHANGES from + to −

④ REPEAT STEP 2 using [-2.2, -2.1]

$\left.\begin{array}{l} f(-2.18) = -.2807 \\ f(-2.17) = .0456 \end{array}\right\}$ CHANGES Sign

⑤ $r \approx -2.17$

Ex 5 Solve the following problems.

#104

a) Find k such that x + 2 is a factor of $f(x) = x^4 - kx^3 + kx^2 + 1$.

$$\begin{array}{r|rrrrr} -2 & 1 & -K & K & 0 & 1 \\ & & -2 & 2K+4 & -6K-8 & 12K+16 \\ \hline & 1 & -K-2 & 3K+4 & -6K-8 & \bigcirc\!\!\!\!\!\!\;12K+17 \end{array}$$

set = 0

$12K + 17 = 0$

$\boxed{K = -17/12}$

b) If -2 is one solution of the equation $x^3 + 5x^2 + 5x - 2 = 0$, what the sum of the remaining solutions?

$$\begin{array}{r|rrrr} -2 & 1 & 5 & 5 & -2 \\ & & -2 & -6 & 2 \\ \hline & 1 & 3 & -1 & 0 \end{array}$$

$x^2 + 3x - 1 \Rightarrow$ USE QUADRATIC formula

$x = \dfrac{-3 \pm \sqrt{(3)^2 - 4(1)(-1)}}{2(1)} = \dfrac{-3 \pm \sqrt{13}}{2}$

Their sum: $\dfrac{-3+\sqrt{13}}{2} + \dfrac{-3-\sqrt{13}}{2} = \dfrac{-6}{2} = -3$

The sum of the remaining solutions is -3.

Ex 6 Is $\frac{2}{3}$ a zero of $f(x) = x^7 + 6x^5 - x^4 + x + 2$? Explain you answer.

$$\begin{array}{r|rrrrrrrr} \frac{2}{3} & 1 & 0 & 6 & -1 & 0 & 0 & 1 & 2 \\ & & 2/3 & 4/9 & 110/27 & 178/81 & \frac{356}{243} & 712/729 & 2882/2187 \\ \hline & 1 & 2/3 & 58/9 & \frac{39}{27} & \frac{178}{81} & \frac{356}{243} & \frac{1441}{729} & \frac{7252}{2187} \end{array}$$

x = 2/3 yields a remainder of 7250/2187. Therefore it is NOT A zero

3.7 Ex 1

A 5th degree polynomial has zeros $4, 6i$, AND $(2-i)$. What are the remaining zeros?

$$\boxed{-6i \text{ AND } (2+i)}$$

Remember conjugates?

3.7 Ex 2

Form a polynomial in standard form. Degree $= 5$

Zeros are $2,\ 1+2i,\ -5i$.

other 2 zeros: $1-2i,\ 5i$

So, factors are: $f(x) = (x-2)(x-(1+2i))(x-(1-2i))(x-5i)(x+5i)$

Rewrite: $f(x) = (x-2)\underline{((x-1)-2i)((x-1)+2i)}\,\underline{(x-5i)(x+5i)}$

(group $(x-1)$)

$f(x) = (x-2)\left[(x-1)^2 - 4i^2\right](x^2 - 25i^2)$

$f(x) = (x-2)\left[x^2 - 2x + 1 - 4(-1)\right](x^2 - 25(-1))$

$f(x) = (x-2)(x^2 - 2x + 5)(x^2 + 25)$

Use "box" technique

- like punnet square

	x^2	$-2x$	-5
x	x^3	$-2x^2$	$-5x$
-2	$-2x^2$	$4x$	$+10$

$\Longrightarrow x^3 - 4x^2 - 9x + 10$

$f(x) = (x^3 - 4x^2 - 9x + 10)(x^2 + 25)$

	x^3	$-4x^2$	$-9x$	$+10$
x^2	x^5	$-4x^4$	$-9x^3$	$10x^2$
25	$25x^3$	$-100x^2$	$-225x$	250

$$\boxed{f(x) = x^5 - 4x^4 - 16x^3 - 90x^2 - 225x + 250}$$

$\boxed{\text{Ex3}}$ $(1+3i)$ is a zero. Find the remaining zeros.

$$g(x) = x^4 - 7x^3 + 14x^2 - 38x - 60$$

$$\frac{p}{q} = \frac{\text{FACTORS OF 60}}{\text{FACTORS OF 1}} \Rightarrow \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 12, \pm 15, \pm 20, \pm 30, \pm 60$$

USE TRIAL AND ERROR UNTIL YOU GET ONE!

$$\begin{array}{r|rrrrr} -1 & 1 & -7 & 14 & -38 & -60 \\ & & -1 & 8 & -22 & 60 \\ \hline & 1 & -8 & 22 & -60 & \vert\ 0 \end{array}$$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 22 & -60 \\ & & 6 & -12 & 60 \\ \hline & 1 & -2 & 10 & \vert\ 0 \end{array}$$

THE REMAINING REAL ZEROS: $x = -1$, $x = 6$

THE OTHER IMAGINARY ZERO: $(1-3i)$

Ex 4 b $f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65$

possible zeros: $\dfrac{\text{FACTORS OF } 65}{\text{FACTORS OF } 2} = \begin{array}{c|c|c|c} \pm 1 & \pm 5 & \pm 13 & \pm 65 \\ \hline \pm 1 & \pm 2 & & \end{array}$

$\pm 1, \pm 5, \pm 13, \pm 65, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm \frac{13}{2}, \pm \frac{65}{2}$

$$\begin{array}{r|rrrr|r} 5 & 2 & 1 & -35 & -113 & 65 \\ & & 10 & 55 & 100 & -65 \\ \hline & 2 & 11 & 20 & -13 & 0 \end{array}$$

$x = 5$ is A zero,
$(x-5)$ is A factor.

$$\begin{array}{r|rrr|r} \frac{1}{2} & 2 & 11 & 20 & -13 \\ & & 1 & 6 & 13 \\ \hline & 2 & 12 & 26 & 0 \end{array}$$

$x = \frac{1}{2}$ is a zero
$(x - \frac{1}{2})$ is A factor
$\Rightarrow (2x-1)$ is A factor

$2x^2 + 12x + 26 = 0$

$2(\underbrace{x^2+6x+13}) = 0$

Doesn't factor.

USE QUADRATIC FORMULA.

$A = 2$
$B = 12$
$C = 26$

$$x = \frac{-12 \pm \sqrt{12^2 - 4(2)(26)}}{2(2)} = \frac{-12 \pm \sqrt{-64}}{4} = \frac{-12 \pm 8i}{4}$$

$\boxed{x = -3 \pm 2i}$ THESE ARE COMPLEX ZEROS

$\therefore \; (x - (-3+2i))$ AND $(x - (-3-2i))$ ARE FACTORS

$$\boxed{f(x) = (x-5)(2x-1)(x^2+6x+13)}$$